ABSTRACT

The paper describes a game-theoretic framework and a computational algorithm for feasibility evaluation of automotive powertrains with storage elements in terms of fuel economy and emissions performance. The game-theoretic framework allows to handle various time-dependent uncertainties, including uncertainties in the drive cycle. In particular, an important issue in the prior approaches to this problem, the drive cycle dependence of the optimal policies, is alleviated. Within the basic framework, it is also possible to generate implementable operating policies that specify powertrain actuator settings as functions of engine operating conditions and states of the storage elements. We illustrate the procedure using an example powertrain with a Direct Injection Stratified Charge engine and an aftertreatment system consisting of a Three Way Catalyst and a Lean NOx Trap.

1 Introduction

This paper develops a methodology for assessing optimal emission constrained fuel economy of powertrain systems with storage elements. As it has been pointed out in (Kolmanovsky et al., 1999), modern powertrain systems may include energy or emission storage elements for which the determination of optimal operating policies and feasibility evaluation in terms of meeting fuel economy and emission targets is substantially more complex than for conventional powertrains.

In particular, this assessment involves dynamic optimization of the trajectories of the powertrain variables over a drive cycle. Specific examples include (Kang et al., 1999), where a direct injection stratified charge engine with a Lean NOx Trap (storage mechanism of NOx and O2) is treated and the objective is to determine the optimal trajectories of the air-to-fuel ratio, EGR rate, spark timing, injection timing and fueling rate to attain best emission constrained fuel economy. Another example is treated in references (Brahma et al., 2000; Lin et al.) where a hybrid electric powertrain is considered and the objective is to determine the optimal power split between the electric motor and the engine at each instant on the drive cycle to maximize the fuel economy. The storage mechanism in that application is the energy storage in the battery. In both of these case studies, the dynamic programming algorithm was used to compute the optimal trajectories for the given drive cycle. Related problems of cold-start optimization have been considered in (Cohen et al., 1984; Sun and Sivashankar, 1998) where the storage mechanism involves heat energy storage in Three-Way-Catalyst (TWC) before the light-off.

The dynamic programming based solutions can be used to determine the best emissions constrained fuel economy and its sensitivity to various powertrain parameters so that appropriate powertrain parameter selection and target cascading decisions can be facilitated. They also offer useful insights into optimal behavior of the powertrain on selected drive cycles. At the same time, they suffer from several limitations: (i) The optimal trajectories depend on time and are drive cycle-dependent so they cannot be easily incorporated into an implementable control strategy; (ii) The time-dependent uncertainties either in powertrain parameters/operation or in the drive cycle are not explicitly accounted for. The first issue is fundamental to systems with storage elements in that the optimal trajectories on the drive cycle explicitly depend on time. For powertrain systems without storage elements, the situation is much simpler (Rao et al., 1979). The optimal settings for actuator and other powertrain variables can be obtained as functions of operating conditions (such as engine speed and engine torque) without explicit dependence on
time. When evaluated at the operating conditions that occur on the
drive cycle, these operating strategies exactly reproduce the optimal
drive cycle trajectories.

In this paper we address the above issues for powertrains with
storage elements and demonstrate how they may be resolved using
the game-theoretic framework (Başar and Olsder, 1982; Kravoski
and Subbotin, 1977; Vorob’ev, 1977). Specifically, the effect of the
uncertainties is represented by the actions of the first player while
the effect of the operating strategy by the actions of the second
player. The second player wishes to minimize a performance index
that reflects control objectives (e.g., minimum emission constrained
fuel consumption) while the first player wishes to maximize either
this, or more generally, a different performance index.

The main focus in this paper is on the treatment of drive cycle
uncertainties and generation of implementable powertrain operating
policies. The Stackelberg feedback equilibrium concept in dynamic
games (Başar and Olsder, 1982; Stackelberg, 1952; Vorob’ev, 1977)
is particularly suitable in this situation. The first player acts as a
leader and at each discrete time instant defines the operating condi-
tions; the second player acts as a follower and provides a rational re-
duction to this action of the first player by specifying optimal settings
for powertrain variables. The game is repeated the next time instant
and so on. Note that the Stackelberg equilibrium concept is different
from a more familiar Nash equilibrium concept when both players
announce their actions simultaneously. As it will be also demon-
strated, the Stackelberg equilibrium solutions can be used to define
implementable powertrain operating policies that prescribe settings
for powertrain actuators and other powertrain variables as functions
of the operating conditions (engine speed and engine torque) and the
values of the states of the storage elements.

The paper is organized as follows. In Section 2 we review the
notion of Stackelberg equilibrium applied initially to static games,
and in Section 3 we apply it to discrete-time dynamic games. The
relations between the Stackelberg equilibrium, Nash equilibrium,
min-max and max-min equilibria are considered in detail and an
approach to treating unmeasured time-dependent uncertainties simul-
taneously with measured uncertainties is outlined. The unmeasured
uncertainties may represent the effects of unknown powertrain pa-
terms or modeling errors. In Section 4 we review a computational
algorithm and in Section 5 we report on an application to a power-
train based on a direct injection stratified charge (DISC) engine.

2 Stackelberg Equilibria in Static Games

Consider a two-person game with sets \( \mathcal{U} \) and \( \mathcal{W} \) of admissible
strategies. For computational efficiency/robustness reasons, that are
especially compelling in a more general dynamic game setting (see
the following section) the selection of an optimal strategy is best
done from a finite set of possibilities. For example, restricting the
search to a reasonable finite set of strategies was the key in making
the dynamic programming approach of (Kang et al., 1999) computa-
tionally tractable. Consequently, both \( \mathcal{U} \) and \( \mathcal{W} \) are assumed to
be finite sets (i.e., they consist only of a finite number of elements).

The players will be identified with their strategies \( u \in \mathcal{U} \) and
\( w \in \mathcal{W} \), respectively. The Player \( u \) wishes to minimize a given cost
function \( f_1(u, w) \) while the Player \( w \) wishes to minimize another
given cost function \( f_2(u, w) \). Both cost functions are known to each
of the players.

We assume that Player \( w \) (the leader) selects a strategy \( w \in \mathcal{W} \)
first and announces the decision to Player \( u \) (the follower). The
Player \( u \) then reacts by picking some \( u \in \mathcal{U} \). The set of rational
reactions of Player \( u \) to a given \( w \in \mathcal{W} \) is

\[
\hat{U}(w) = \{ \hat{u} \in U : f_1(\hat{u}, w) \leq f_1(u, w), \forall u \in U \}.
\]

If the set \( \hat{U}(w) \) is not a singleton set, the Player \( w \) cannot predict
which rational reaction Player \( u \) will use. Hence, Player \( w \) tries to
select a strategy \( w^* \) that satisfies

\[
\max_{\hat{u} \in \hat{U}(w)} f_2(\hat{u}, w^*) = \min_{w \in \mathcal{W}} \max_{\hat{u} \in \hat{U}(w)} f_2(\hat{u}, w). \quad (1)
\]

The strategy \( w^* \) is referred to as the Stackelberg strategy of the
leader.

Suppose now that the value of \( f_2(u, w) \) for \( u \in \hat{U}(w) \) does not
depend on \( u \). This would be the case, for example, in a zero sum
game, where \( f = f_1 = -f_2 \) and thus the players have completely
opposing interests. In this situation, the Stackelberg strategy can be
defined in a different way. Indeed, consider any selection

\[
U(w) = \{ u \in U : \min_{w \in \mathcal{W}} f_2(u, w) \leq f_2(\tilde{u}, w), \forall u \in \hat{U}(w) \}. \quad (2)
\]

Then, if \( w^* \) is a Stackelberg strategy, we have

\[
f_1(U(w), w) \leq f_1(u, w), \quad f_2(U(w^*), w^*) \leq f_2(U(w), w), \forall u, w. \quad (3)
\]

So, a Stackelberg strategy \( w^* \) and a selection \( U(w) \) issues a pair
\((u^*, w^*)\) where \( u^* = U(w^*) \), that is called a Stackelberg equilibrium
pair. Conversely, every selection \( U(w) \) allows one to reconstruct the
Stackelberg strategy \( w^* \). If \( f = f_1 = -f_2 \), \( (3) \) can be modified to the form

\[
f_1(U(w), w) \leq f(U(w^*), w^*) \leq f(u, w^*), \forall u, w.
\]

3 Feedback Stackelberg Equilibria in Dynamic Games

Next we consider a dynamic two-person game, where the evolu-
tion of the state variable is described by the discrete-time equations

\[
x(t + 1) = f(x(t), u(t), w(t)), \quad t = 0, 1, \ldots, T - 1, \quad (4)
\]
with a finite final time $T$ and a specified initial state $x(0) = x_0 \in X$. Here $u(t)$ is the control of Player $u$ and $w(t)$ is the control of Player $w$ at the discrete time instant $t$. The sets of admissible controls $U \subset R^n, W \subset R^m$ are independent of $t$.

The goal of Player $u$ is to minimize a stage-additive cost function

$$J(u(\cdot), w(\cdot)) = \sum_{t=0}^{T-1} H(x(t), u(t), w(t), t), \quad (5)$$

while the goal of Player $w$ is to maximize $J$. Here $H$ is a continuous function of the four arguments: $x, u, w,$ and $t$. The time-dependent costs may arise, for example, in a situation when we wish to penalize the deviation of $w$ from a prescribed drive cycle trajectory that is a function of time.

We assume that, when selecting $u(t)$ at a time instant $t$, Player $u$ knows the current state of the system $x(t)$ and also knows the control $w(t)$ already selected by Player $w$. The Stackelberg equilibrium concept applied to this dynamic game at every state $t$ (Simaan and Cruz, 1973) leads to the following definition.

Definition. A pair of sequences $(u^{*}(\cdot), w^{*}(\cdot))$ constitutes a feedback Stackelberg equilibrium pair in the dynamic antagonistic game with Player $w$ as the leader and Player $u$ as the follower if:

$$J(u^{*}(\cdot), w^{*}(\cdot)) = \min_{w(0)} \max_{w(1)} \min_{w(2)} \ldots \max_{w(T-1)} \min_{w(T)} J(u(\cdot), w(\cdot)). \quad (6)$$

The value $J(u^{*}(\cdot), w^{*}(\cdot))$ is called the value of the game.

As in the optimal control problems, the optimal “cost-to-go” in the game problems can be described using a Bellman function. The Bellman function is defined as

$$V(t, x) = \max_{w(t)} \min_{u(t)} \max_{w(t+1)} \ldots \max_{w(T-1)} \min_{w(T)} \sum_{\tau=t}^{T-1} H(x(\tau), u(\tau), w(\tau), \tau), \quad (7)$$

with a domain $t = 0, 1, \ldots, T - 1, x \in X$, where $x(\tau)$ evolves according to equation $(4)$ with the initial condition $x(t) = x$. If a feedback Stackelberg equilibrium exists, it has to satisfy $V(0, x_0) = J(u^{*}(\cdot), w^{*}(\cdot))$. We now demonstrate how a feedback Stackelberg equilibrium can be constructed. As a by-product of this construction, the existence of such equilibria is also established.

Consider a time instant $(T - 1)$ and let us treat $x \in X$ as a possible state $x'(T - 1)$. If Player $w$ announces some $w$ at time $T - 1$, Player $u$ would have to select a control from $\widehat{U}(T - 1, x, w) \triangleq \{u : u \in \arg \min_{u \in U} H(x, u, w, T - 1)\}$. Assuming this rational reaction of Player $u$, Player $w$ has to select the control from $\widehat{W}(T - 1, x) \triangleq \{w : w \in \arg \max_{w \in W} \min_{u \in U} \{H(x, u, w, T - 1)\}\}$. It is clear that $V(T - 1, x) = \max_{w \in W} \min_{u \in U} \{H(x, u, w, T - 1)\}$. Now, if we have determined $V(t + 1, \cdot)$ for some $t \in \{0, 1, \ldots, T - 2\}$, we can compute $V(t, \cdot)$ using a recursion:

$$\hat{U}(t, x, w) = \{\hat{u} : \hat{u} \in \arg \min_{u \in U} \{H(x, u, w, t) + V(t + 1, f(x, u, w))\}\}$$

$$(V(t), w) = \min_{w \in W} \{H(x, u, w) + V(t + 1, f(x, u, w))\}$$

$$\hat{W}(t, x) = \{\hat{w} : \hat{w} \in \arg \max_{w \in W} \{V(t, x, w)\}\}$$

$$V(t, x) = \max_{w \in W} \{V(t, x, w)\}$$

$t = 0, 1, \ldots, T - 2$. By construction, they yield the following result:

Proposition. In a two-person dynamic antagonistic game with Player $w$ as the leader and Player $u$ as the follower, there exists a feedback Stackelberg equilibrium pair. Any selections $U(t, x, w) \in \hat{U}(t, x, w), W(t, x) \in \hat{W}(t, x), t = 0, 1, \ldots, T - 2$. By construction, they yield the following result:

$$w^{*}(t) = W(t, x^{*}(t)), u^{*}(t) = U(t, x^{*}(t), w^{*}(t)),$$

$$x^{*}(t + 1) = f(x^{*}(t), u^{*}(t), w^{*}(t)), t = 0, 1, \ldots, T - 1, x^{*}(0) = x_0.$$  \quad (9)

Conversely, for any Stackelberg equilibrium pair $(u^{*}(\cdot), w^{*}(\cdot))$, there exist selections $U(t, x, w), W(t, x)$ such that (9) holds.

We see that the feedback Stackelberg equilibrium pair has the following property: at any time instant $t$, $(u^{*}(t), w^{*}(t))$ is a Stackelberg equilibrium pair in a static antagonistic game with Player $w$ as the leader and Player $u$ as the follower, with the cost function $J(u, w) = H(x(t), u, w, t) + V(t + 1, f(x(t), u, w))$. Summarizing, we can formulate an important property of the function $V(t, x)$ which serves as the basis of the numerical procedure for finding the feedback Stackelberg equilibria.

Proposition. The function $V(t, x)$ satisfies the equation

$$V(t, x) = \max_{w \in W} \{H(x, u, w, t) + V(t + 1, f(x, u, w))\}. \quad (10)$$

To shed light on the properties of the Stackelberg equilibrium strategy, let us recall several other rationales used in the game theory. A strategy $w^{*_M}(\cdot)$ is called the max-min strategy of Player $w$ if

$$J_u \triangleq \min_{u(\cdot)} J(u(\cdot), w^{*_M}(\cdot)) = \min_{u(\cdot)} J(u(\cdot), w^{*}(\cdot)).$$

This strategy is best for Player $u$ in the game, because Player $w$ has to announce his strategy $w^{*}(\cdot)$ at the initial time instant for all time instances $t = 0, 1, \ldots, T - 1$ ahead. Defining the min-max strategy
$u_\delta^*(\cdot)$ in a usual way

$$J^* \triangleq \max_{w(\cdot)} \min_{u(\cdot), w(\cdot)} \mathcal{J}(u(\cdot), w(\cdot))$$

that corresponds to the situation when Player $u$ has to announce his strategy $u(\cdot)$ at the initial time instant for the all time instances $t = 0, 1, \ldots, T - 1$ ahead, without knowing about a strategy $w(\cdot)$. It is easy to see that

$$J \leq J(u^*(\cdot), w^*(\cdot)) \leq J^*.$$ \hspace{1cm} (11)

Note that if $J_\delta = J^*$, then $(u^*(\cdot), w^*(\cdot))$ define a Nash equilibrium in the game. In general, Nash equilibria exist only under special assumptions while the existence and determination of feedback Stackelberg equilibria can be done under much weaker assumptions.

The feedback form of the Stackelberg equilibrium strategies $U(t, x, w) \in \tilde{U}(t, x, w)$ is attractive as far as implementation is concerned. It is easy to see that the difference in performance between a time-invariant control strategy $u(t) = U(0, x(t), w(t))$ and the optimal strategy $u(t) = U(t, x(t), w(t))$ is not that great, except for $t$ close to the end-horizon, $T$. Then, $u = U(0, x, w)$ can be used as an implementable control strategy to determine a near optimal reaction to $w$, given a known state $x$.

Finally, consider a situation when Player $u$ at time $t$ knows control $w(t)$ only in part. In the powertrain assessment problem, this situation occurs when there are both uncertainties in the drive cycle and in the parameters of either the powertrain itself or of the powertrain model. Considering $w(t) \in R^M$, suppose Player $w$ announces a part of the control $w(t) = (w_1(t), w_2(t), \ldots, w_m(t)), m < M$, whereas $\tilde{w}(t) = (w_m(t), \ldots, w_M(t))$ is concealed from Player $u$. Here an equilibrium pair $(u^+(\cdot), w^+(\cdot))$ may be defined as a control sequence that achieves the equality

$$\mathcal{J}(u^+(\cdot), w^+(\cdot)) = \min_{u(t)} \max_{w(t)} \mathcal{J}(u(\cdot), w(\cdot))$$

assuming that $u(t) \in \tilde{U}(t), w(t) = (\tilde{w}(t), \tilde{w}(t)) \in \tilde{U}$ for all $t = 0, 1, \ldots, T - 1$. Clearly,

$$J_\delta \leq \mathcal{J}(u^+(\cdot), w^+(\cdot)) \leq \mathcal{J}(u^+(\cdot), w^+(\cdot)) \leq J^*.$$  

4 Computations

4.1 Preliminaries

The backward-in-time iterations are used to calculate the Bellman function $V(t, x)$. The computational procedure is analogous to the Dynamic Programming algorithm but it applies to a more general, game problem.

First, it is necessary to quantize the values of the state variable $x$. Suppose that the state trajectory $x(t)$ remains confined to a compact set $S$ no matter what controls $u(t) \in \tilde{U}$ and $w(t) \in \tilde{W}$, $t = 0, 1, \ldots, T - 1$ are. The admissible strategy sets $\tilde{U}$ and $\tilde{W}$ are assumed to be finite. Given $h > 0$, we can construct a mesh $\{x_1, x_2, \ldots, x_K\} \in S$ satisfying the inequality $\min_{k=1,\ldots,K} \|x - x_k\| \leq h, \forall x \in S$.

Suppose that the functions $H(x, u, w, t)$ and $f(x, u, w)$ are continuous in their arguments. The results in (Vorob’ev, 1977) imply the continuity in $x$ of the function $\min_{w \in \tilde{W}} \{H(x, u, w, t)\}$, i.e.,

$$V(T - 1, x).$$

By induction in $t$ we can show that the Bellman function $V(t, x)$ is continuous in $x$ for any $t$.

Define an approximation $V_h(t, x)$ for $V(t, x)$ by

$$V_h(t, x) = \max_{w(t) \in \tilde{W}} \min_{u(t) \in \tilde{U}} \{H(x, u, w, t)\} + V_h(t + 1, R[f(x, u, w)]) \quad t = 0, 1, \ldots, T - 2, k = 1, 2, \ldots, K.$$  \hspace{1cm} (11)

Here $R[x] \in \arg \min_{1 \leq k \leq K} \|x - x_k\|$. It can be shown that

$$V_h(t, x_k) \rightarrow V(t, x_k) \text{ as } h \rightarrow 0, t = 0, \ldots, T - 1.$$  

This result substantiates the use of the quantization procedure to determine approximate solutions to the game problem.

4.2 Algorithm

We now summarize the algorithm for calculating an approximate feedback Stackelberg equilibrium.

**Backward pass**

**Step T-1** Calculate

$$V_h(T - 1, x),$$  \hspace{1cm} (11)

**Step t, (s=T-2, T-3, \ldots, 0)** For each $x_k, k = 1, 2, \ldots, K, w \in \tilde{W}$, determine

$$U_h(t, x_k, w) \in \arg \min_{u(t)} \{H(x_k, u(t), w) + V_h(t + 1, R[f(x_k, u(t), w)])\}$$

$$W_h(t, x_k) \in \arg \min_{w(t) \in \tilde{W}} \{H(x_k, U_h(t, x_k, w), t) + V_h(t + 1, R[f(x_k, U_h(t, x_k, w), w)])\}$$

$$V_h(t, x_k) = H(x_k, U_h(t, x_k), w) + V_h(t + 1, R[f(x_k, U_h(t, x_k), w)])$$

where $\tilde{w} = W_h(t, x_k), \tilde{u} = U_h(t, x_k, \tilde{w})$.
Forward pass

Step 0 Given $x_0$, calculate
\[
\begin{align*}
  w^*(0) &= W_h(0, R[x_0]) \\
  u^*(0) &= U_h(0, R[x_0], w^*(0)) \\
  x^*(1) &= f(x_0, u^*(0), w^*(0))
\end{align*}
\]

Step $t$, $t=1, 2, \ldots, T-1$ Calculate
\[
\begin{align*}
  w^*(t) &= W_h(t, R[x^*(t)]) \\
  u^*(t) &= U_h(t, R[x^*(t)], w^*(t)) \\
  x^*(t+1) &= f(x^*(t), u^*(t), w^*(t))
\end{align*}
\]

This computational scheme was implemented in Matlab. Matlab uses built-in algorithms that are optimized for performing operations on vectors and matrices. Hence, the utilization of vector and matrix updates is preferred to performing algorithm steps on element-by-element basis in multiple nested “for” loops. Specifically, since sets $\mathcal{U}$ and $\mathcal{W}$ are finite we may pre-compute the quantities $H(x_k, u_l, w_r, t)$ for all $x_k \in S$, $u_l \in \mathcal{U}$, $w_r \in \mathcal{W}$, $t = 0, \cdots, T-1$ and store them in an array $\hat{H} = [H(x_k, u_l, w_r, t)]_{k,l,r,t}$. Similarly, if $x_k = R[f(x_k, u_l, w_r)] \in S$, we can store indices $\hat{k} = k(k, l, r)$ in another array, $\hat{V}_h(t+1) = V_h(t+1, R[f(x_k, u_l, w_r)])_{k,l,r}$. This procedure significantly reduces the computational times.

5 Application to DISC Engines

In this Section, we apply the game-theoretic methodology to the optimization of the fuel economy and emissions in a powertrain with an direct injection stratified charge (DISC) engine, see Figure 1. See (Druzhinina et al., 1999) for a concise description of DISC engine operation while references (Druzhinina et al., 1999; Kolmanovsky et al., 2000; Sun et al., 1999; Sun et al., 2001) discuss various aspects of DISC engine modeling and control.

Because of the mostly lean operation, a DISC engine requires, in addition to a Three-Way Catalyst (TWC), another TWC referred to as a Lean NOx Trap (LNT). The LNT and TWC are dynamic storage devices which store NOx during lean operation and can be purged by running the engine rich for a few seconds. During the purge the NOx stored in the LNT is released and converted to nitrogen and water.

Following (Kang et al., 1999), we consider a two-state model for the system, where the states are the fraction of the TWC oxygen storage capacity filled with oxygen and a fraction of NOx storage capacity of the LNT filled with NOx. A quasi-static engine model, described in (Sun et al., 1999), and a static model for the aftertreatment temperature have been assumed so that the only dynamics are due to the storage mechanism in the aftertreatment system. The dynamics, in discrete-time form, can be summarized by the following difference equations:

\[
x(t+1) = f(x(t), v(t), w(t)),
\]

where $v(t)$ is the vector of engine control variables at time $t$ (combustion mode, spark timing, fuel quantity, injection timing, EGR rate, air-to-fuel ratio) and $w(t)$ is the vector of engine operating conditions at time $t$ (engine speed $N(t)$ and engine brake torque $T_b(t)$).

In (Kang et al., 1999) the problem of fuel economy/emission assessment for a DISC engine system without uncertainties is considered. It is shown that this problem reduces to the determination of the policies that minimize the following cost:

\[
J = \sum_{t=0}^{T} \left[h_f(x(t), v(t), w(t)) + \mu h_{\text{nox}}(x(t), v(t), w(t))\right],
\]

where $T$ is the duration of the drive-cycle over which the assessment is conducted, $h_f$ determines the instantaneous fuel consumption, $h_{\text{nox}}$ determines the instantaneous tailpipe NOx emissions flow rate. The trade-off plot of total fuel consumption versus total NOx emissions resulting as the parameter $\mu$ varies has been shown to yield important information with respect to system feasibility in terms of meeting emission standards, assessment of emission constrained fuel economy, dependence of these conclusions on system parameters and, finally, the impact of architectural decisions in the control strategy on the emission constrained fuel economy. The optimal policies obtained in (Kang et al., 1999) depend on time and state of the aftertreatment system; they are optimized for the particular drive cycle.

We now illustrate the game-theoretic solution to this problem that does not assume a fixed drive cycle and provides optimal, in a feedback Stackelberg equilibrium sense, policies $U(t, x, w)$. These policies depend on engine operating conditions $w$, state of the aftertreatment system $x$ and time $t$. Our numerical experience indicates that if the end-horizon $T$ is sufficiently large (but still
much smaller than the duration of standard drive cycles), \( \hat{U}(x, w) = U(0, x, w) \) may be adopted as a time-invariant control policy which prescribes engine control variable settings as functions of the state of the aftertreatment system and of the speed/torque operating point only. If \( T \) is sufficiently large the behavior and performance due to \( \hat{U}(x, w) \) are very close to those due to \( U(t, x, w) \).

We assume that \( w(t) \) can only take one of a finite number of values, \( w(t) = [w_1(t), w_2(t)] \in W, \) corresponding to the centers of the cells in the standard rectangular cell quantization of the speed-torque domain, see (Kolmanovsky et al., 1999) and Figures 3.4. Specifically, if \( N_s = [N_1, \cdots, N_m], \) \( \tau_s = [\tau_1, \cdots, \tau_m] \) are the engine speed and brake torque grid points, then \( w_i(t) \) is the index of \( N(t) \) in \( N_s \) and \( w_2(t) \) is the index of \( \tau(t) \) in \( \tau_s \). We, initially, assume that the engine can be operated using one of four fixed policies \( u(t) \in U = \{1, 2, 3, 4\} \) wherein \( u = 1 \) corresponds to the stratified lean operation, \( u = 2 \) corresponds to the homogeneous lean operation, \( u = 3 \) corresponds to the stoichiometric operation, and \( u = 4 \) corresponds to the rich purge operation. These sub-modes are differentiated by air-fuel ratio ranges and injection timing (late for stratified, early for the rest). In each of \( u = 1, 2, 3 \) sub-modes, \( v \) is set to minimize the instantaneous, emission-constrained fuel consumption subject to constraints on the range of \( v \), i.e., \( v = v^*(u, w) \). In \( u = 4 \) sub-mode, \( v \) is set to minimize the purge time of LNT.

Note that \( v^*(u, w) \) may not be defined for some \( u, w \). For example, maintaining stratified operation at high torque values may require the intake manifold pressure to be above the ambient, which is not possible. Hence, in this problem the choice of \( u \) is constrained by the choice of \( w \), i.e., \( u \in U(w) \). The theory carries over to this more general case with the notational changes only.

Slightly abusing the notation, by using the same \( f \) as in (12), we obtain the dynamics in the form:

\[
x(t + 1) = f(x(t), u(t), w(t)).
\]  \( \text{(14)} \)

The direct application of (13) leads to an unrealistic \( w \)-trajectory, which is restricted entirely to high speed high torque conditions. The idea then is to consider a modified cost based on (5) with

\[
H = (h_f(x, v^*(u, w), w) + \mu h_{\text{nox}}(x, v^*(u, w), w)) \cdot S(w),
\]  \( \text{(15)} \)

where \( S(w) \) is a scaling function. In this example, we employed scaling by the inverse of the engine indicated power, \( P_{\text{ind}}(w) \), i.e., \( S(w) = 1/P_{\text{ind}}(w) \).

The results of numerical implementation are now reported. We used a uniform quantization \( x_k = (x_{1i}, x_{2j}) \), \( i, j = 1, \ldots, 11 \) of the domain \( S = [0, 1] \times [0, 1] \). The optimization with the end-horizon \( T = 100 \) sec was performed for several values of \( \mu \) and the optimal control policies \( u^*(t) = U(t, x(t), w(t)) \) and the corresponding worst drive cycle \( w^*(t) = W(t, x(t)) \) were obtained. The corresponding time invariant policies, defined as \( \hat{U}(x, w) = U(0, x, w) \), were evaluated over the European drive cycle (NEDC), which lasts for 1178 sec, and a trade-off curve of fuel consumption in mpg versus total NOx emissions in g/km was obtained, see Figure 2. By intersecting the trade-off curve with the legislated emission level corresponding to Stage IV European emission standard (see the dashed line in Figure 2), the value of \( \mu = 19 \) was determined. Figures 3 and 4 illustrate the policies \( \hat{U}([1, 0], w) \) and \( \hat{U}([1, 1], w) \) for this value of \( \mu \). The first of these example policies is used when TWC storage is full while LNT NOx storage is empty. The second of these example policies is utilized when TWC oxygen storage and LNT NOx storage are full. The results are consistent with the expectations: if LNT is empty, its conversion efficiency is high and we can take advantage of either stratified lean (\( u = 1 \)) or homogeneous lean (\( u = 2 \)) operation at low speeds and torques to reduce fuel consumption. At the cell corresponding to minimum speed-minimum torque conditions (i.e., near idle) the optimal policy is, however, stoichiometric (\( u = 3 \)). Most likely this is caused by the fact that near idle the LNT temperature cannot be maintained within an acceptable window for LNT to have sufficient conversion efficiency.
behavior for $t \leq 75$ sec. For $t > 75$ sec, the policy $U$ allows LNT to fill freely while $\tilde{U}$ enforces LNT purges intermittently with periods of stratified lean operation. The optimal cost values are very close in these two cases: 34.5796 for $U$ and 35.5224 for $\tilde{U}$.

The evaluation of the policy $\tilde{U}$ on a simulated European drive cycle is demonstrated in Figure 7. All four operating modes (stratified lean, homogeneous lean, stoichiometric, and purge) are exercised by the policy $\tilde{U}$. There are 13 LNT purges over the drive cycle.

Comparing the fuel economy and emissions curves in Figure 2 with the best achievable ones over the drive cycle in (Kang et al., 1999), we note that there is about 3 mpg deficiency. This gap can be reduced if we allow more sub-modes within the homogeneous lean and stratified modes. More sub-modes were, in fact, used in (Kang et al., 1999). The trade-off curve generated for the case when we assume 3 sub-modes of the lean homogeneous and 3 sub-modes of the stratified mode, generated to minimize a weighted sum of fuel consumption and NOx emissions in each speed-torque cell and in each mode for three different values of the weight, is shown in Figure 8. There is about 2 mpg improvement in best achievable fuel consumption under NOx emissions constraint as compared with the results in Figure 2.

6 Conclusions

The paper described a game-theoretic methodology and a computational algorithm for feasibility evaluation and generation of implementable policies for automotive powertrain with storage elements. By considering optimal powertrain operating policies against worst possible drive cycle, it has been shown how the drive cycle dependence of the optimal policies can be alleviated. We have also shown how implementable time-invariant policies that perform
closely to the time-varying optimal ones can be generated with a simple procedure. The procedure has been illustrated on an example of a powertrain with a direct injection stratified charge engine, Three Way Catalyst and a Lean NOx Trap.

There are several issues that warrant further investigation. One issue has to do with various procedures for selecting the cost (5). These procedures can potentially affect in significant ways the resulting operating policies. Another issue has to do with the fact that some engine management systems may not support a full-blown form of dependence of the operating policy on state variables. They may only accommodate a more limited form of the dependence such as different air-to-fuel ratio values in the lean mode and in the purge mode, but not the air-to-fuel ratio variations within the lean mode. One approach in this case, is to use a fixed structure optimization (Kolmanovsky et al., 1999) as a way of generating implementable operating policies in a desired form.

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REFERENCES


